Vena Contracta

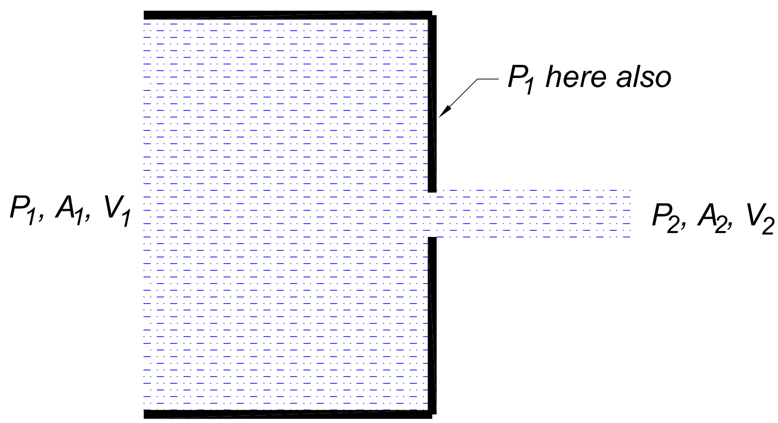
Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(February 16, 2005)

1 Problem

The vena contracta (apparently first described by Torricelli in 1643)[1] is the reduction in the area/diameter of a fluid jet after it emerges from a circular aperture in a pressurized reservoir.[2] Make an estimate of the reduction in area based on a comparison of energy and momentum in the system (Borda, 1766 [2]).

2 Solution

If there were no *vena contracta*, the fluid flow out of a (circular) aperture of area A_2 in the end wall of a (circular) pipe of area A_1 be cylindrical, as sketched in the figure below.



We suppose that area A_1 is large compared to A_2, and that the fluid is incompressible, such that

$$v_1 A_1 = v_2 A_2, \tag{1}$$

where v_1 and v_2 are the average velocities of the fluid flow parallel to the axis of the pipe in the regions of areas A_1 and A_2, respectively. Then,

$$v_1 \ll v_2. \tag{2}$$

We also suppose that the flow velocities are small enough that energy loss due to viscosity can be ignored. Then, Bernoulli's Law holds for points along streamlines,

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2, \tag{3}$$

[1]Torricelli could well have observed the *vena contracta* during his studies of water emerging from holes in tanks, reported on pp. 191-204 of [1], though I don't find explicit mention of this there.

[2]The vena contracta is distinct from the effect of gravity on an (incompressible) fluid jet, such that as the jet falls and picks up speed v, its cross-sectional area A must decrease according to the $vA =$ constant.

where P is pressure and ρ is the (constant) mass density of the fluid. From eqs. (2) and (3) we infer that

$$v_2^2 \approx 2\frac{P_1 - P_2}{\rho}\,. \tag{4}$$

In addition, we consider the momentum in the system. The mass flux is $\rho v A$, so the momentum flux (momentum passing across area A per unit time) is $\rho v^2 A$. The net flux of momentum through a volume bounded by area A_1 on the left and area A_2 on the right is

$$\frac{dp}{dt} = \rho(v_2^2 A_2 - v_1^2 A_1) \approx \rho v_2^2 A_2. \tag{5}$$

This change of momentum is associated with the net force on this volume,

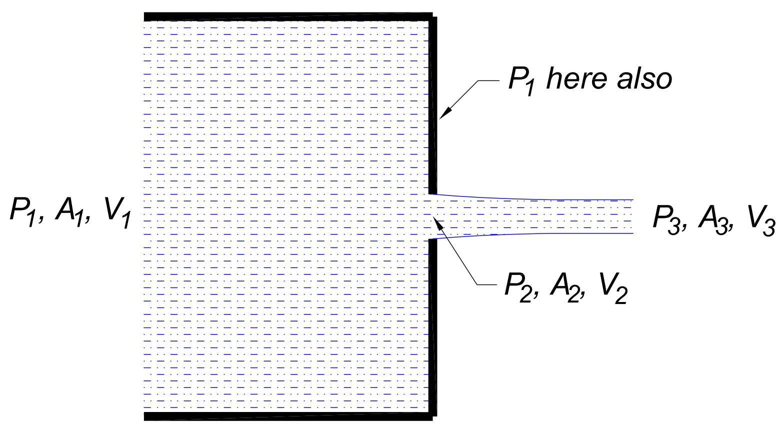
$$F \approx P_1 A_1 - [P_1(A_1 - A_2) + P_2 A_2] = (P_1 - P_2)A_2, \tag{6}$$

noting that the pressure on the wall of area $A_1 - A_2$ is essentially P_1. Then, equating this force to the rate of change (5) of momentum in the fluid, we infer that

$$v_2^2 \approx \frac{P_1 - P_2}{\rho}\,, \tag{7}$$

in contradiction to the result (4) that was based on conservation of energy.

As noted by Borda, this contradiction is resolved in Nature by a contraction of the fluid to area A_3 after it passes through the aperture of area A_2, as sketched below.



The momentum flux is actually

$$\frac{dp}{dt} = \rho(v_2^3 A_3 - v_1^2 A_1) \approx \rho v_3^2 A_3 \approx 2P_1 A_3, \tag{8}$$

according to Bernoulli's equation in the limit that $P_3 \ll P_1$. The force that causes this momentum change is now[3]

$$F \approx P_1 A_1 - [P_1(A_1 - A_2) + P_3 A_3] = P_1 A_2 - P_3 A_3 \approx P_1 A_2. \tag{9}$$

[3]The surface tension balances the force on the surface of the fluid jet due to the difference in pressures inside and outside the jet. Hence, eq. (9) includes no term associated with the difference in areas $A_2 - A_3$.

Altogether, we estimate the *vena contracta* to be

$$A_3 = \frac{A_2}{2}\,. \tag{10}$$

Experiment indicates that the ratio A_3/A_2 is close to 0.64.[4]

A Appendix: Vena Contracta in Two-Dimensional Potential Flow Through a Slot

A comment by Maxwell [4] in 1869 on the *vena contracta* suggested that a more accurate analysis could be based on the velocity potential ϕ that exists when the flow is irrotational ($\boldsymbol{\nabla} \times \mathbf{v} = 0 \Rightarrow \mathbf{v} = -\boldsymbol{\nabla}\phi$). If the fluid is incompressible, such that $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$, the velocity potential satisfies Laplace's equation $\nabla^2\phi = 0$.

Recall that any analytic function $w(z) = \phi + i\psi$ of the complex variable $z = x + iy$ obeys the Cauchy-Riemann equations,

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}\,, \qquad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}\,, \tag{11}$$

which imply that $\nabla^2\phi = 0 = \nabla^2\psi$. That is, any analytic function of a complex variable can, in principle, be related to the velocity potential of some two-dimensional, incompressible, irrotational fluid flow. Furthermore, the curves $\psi(x, y) =$ constant follow streamlines of the potential $\phi(x, y)$ (and *vice versa*).

The case of potential flow through slots in two dimensions was discussed by Helmholtz [5] in 1868, by Kirchhoff [6] in 1869, and the explicit relation of their work to the *vena contracta* was noted by Rayleigh [7] in 1876. Helmholtz [5] argued that if the flow is constrained by planar boundaries in some region of the (x, y) plane (with streamlines along, say, constant y for $x < x_0$) then it is useful to consider an implicit function $z = f(w)$ such as $z = w + e^w = \phi + e^\phi \cos\psi + i(\phi + e^\phi \sin\psi) = x + iy$. This proves to describe fluid flow down a channel (and into an infinite, surrounding reservoir) defined by two parallel planes $(x \le 1, y = \pm\pi)$ that correspond to the streamlines $\psi = \pm\pi$.[5]

Kirchhoff [6] noted that if the fluid flow includes a free surface, on which the pressure is constant, then Bernoulli's equation implies that the velocity of the fluid flow is constant on this surface. This velocity can always be scaled to unity. Then, since $\mathbf{v} = -\boldsymbol{\nabla}\phi = -(\partial\phi/\partial x, \partial\phi/\partial y)$, the scaled flow on a free surface obeys

$$\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 = 1. \tag{12}$$

Furthermore, the Jacobian transformation of area elements tells us that

$$dx\,dy = J\ d\phi\ d\psi = JJ'\ dx\ dy, \tag{13}$$

[4]The *vena contracta* was discussed (but not so named) by Newton in Book II, Prop. 36 of [3], where he reported (p. 333) that $A_3/A_2 = (5/6)^2 = 0.69$.

[5]The same function usefully describes the equipotentials and field lines at the edge of a two-dimensional capacitor, as discussed by Maxwell in sec. 202 of his *Treatise* [8].

so $JJ' = 1$. The Cauchy-Riemann equations for $\phi + i\psi = w(x + iy)$ and $x + iy = w^{-1}(\phi + i\psi)$ allow us to write the Jacobian determinants as

$$J = \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial x} = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2, \qquad \text{and} \qquad J' = \left(\frac{\partial x}{\partial\phi}\right)^2 + \left(\frac{\partial y}{\partial\phi}\right)^2. \tag{14}$$

Then, the scaled flow on a free surface also obeys

$$\left(\frac{\partial x}{\partial\phi}\right)^2 + \left(\frac{\partial y}{\partial\phi}\right)^2 = 1. \tag{15}$$

Kirchhoff [6] extended Helmholtz' technique of functions $w(z)$ that are defined implicitly via knowledge of $z(w)$ by consideration of the derivative dz/dw defined as

$$\frac{dz}{dw} = -f(w) - \sqrt{f^2(w) - 1}, \tag{16}$$

arguing that if along some portion of a streamline ψ = constant the function f is real and $|f|$ less than 1, then $dx/d\phi = Re(dz/dw) = -f$ and $dy/d\phi = Im(dz/dw) = -\sqrt{1 - f^2}$, such that eq. (15) is satisfied, and this portion of the streamline corresponds to a free surface of the fluid flow.

In particular, the function

$$f = e^{-w} = e^{-\phi}\cos\psi - ie^{-\phi}\sin\psi \tag{17}$$

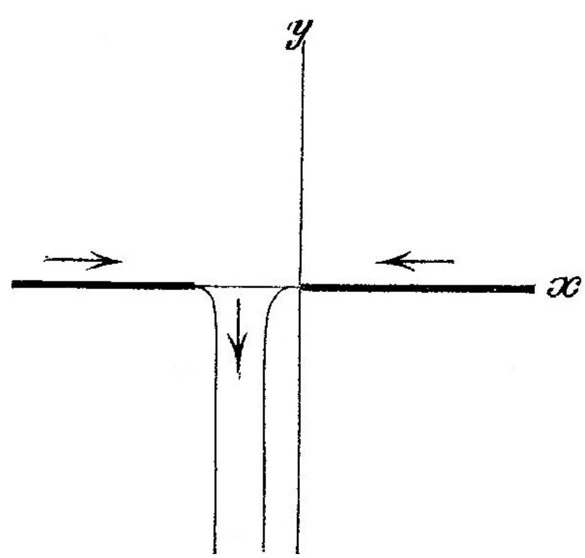
obeys eq. (15) for $\psi = 0$ and $\phi > 0$. The free surface is defined by the equations

$$\frac{dx}{d\phi} = -f = -e^{-\phi}, \qquad \frac{dy}{d\phi} = -\sqrt{1 - f^2} = -\sqrt{1 - e^{-2\phi}} \qquad (\phi > 0, \psi = 0), \tag{18}$$

which integrate to

$$x = e^{-\phi} - 1, \qquad y = \sqrt{1 - e^{-2\phi}} - \ln\left(e^{\phi} + \sqrt{e^{2\phi} - 1}\right) \qquad (\phi > 0, \psi = 0), \tag{19}$$

taking the free surface to begin at the origin, and using Dwight 361.01 after the substitution $s = e^{-\phi}$. On this streamline ($\psi = 0$) but for $\phi < 0$ the derivative $dz/dw = dz/d\phi$ is real and negative, so y stays constant at 0 while x increases from 0 to ∞ as ϕ decreases. The streamlines $\psi = 0$ and π are illustrated in the figure below by Kirchhoff.



Similarly, the streamline $\psi = \pi$ also obeys eq. (15) for $\phi > 0$, such that

$$\frac{dx}{d\phi} = e^{-\phi}, \qquad \frac{dy}{d\phi} = -\sqrt{1 - e^{-2\phi}} \qquad (\phi > 0, \psi = \pi), \tag{20}$$

which integrate to

$$x = -x_0 + 1 - e^{-\phi}, \qquad y = \sqrt{1 - e^{-2\phi}} - \ln\left(e^{\phi} + \sqrt{e^{2\phi} - 1}\right) \qquad (\phi > 0, \psi = \pi). \tag{21}$$

On this streamline $dz/d\phi$ is real and negative for $\phi < 0$, so y is again constant at 0 while x decreases from $-x_0$ to $-\infty$.

Furthermore, on the streamline $\psi = \pi/2$, dz/ϕ is purely imaginary so that x is constant (at $-x_0/2$), and this streamline is the symmetry axis of the flow.

The constant x_0 can be determined by noting that for large negative y the streamlines $0 \le \psi \le \pi$ are all in the y direction, so the corresponding equipotentials of ϕ are at constant y. Then, the velocity $\mathbf{v} = -\boldsymbol{\nabla}\phi$ is constant across the jet, with value unity, such that integrating across the jet at large negative y we find, using the Cauchy-Riemann equations (11),

$$\Delta\psi = \pi = \int_{-x_0+1}^{-1} \frac{\partial \psi}{\partial x}\, dx = -\int_{-x_0+1}^{-1} \frac{\partial \phi}{\partial y}\, dx = v \int_{-x_0+1}^{-1} dx = x_0 - 2, \tag{22}$$

and hence the width of the aperture is $x_0 = \pi + 2$. Thus, we determine the *vena contracta* to be

$$A_{\text{contracted jet}} = \frac{\pi}{\pi + 2} A_{\text{aperture}} = 0.61 A_{\text{aperture}}. \tag{23}$$

According to eq. (19), 90% of the contraction has occurred when $e^{-\phi} = 0.1$, at which $y \approx -2$. That is, 90% of contraction occurs within a distance from the aperture of 0.4 of its width.

B Appendix: Borda's Mouthpiece

Kirchhoff (and earlier, Helmholtz) also considered the function

$$f = -1 - e^{-w} = -1 - e^{-\phi}\cos\psi + i e^{-\phi}\sin\psi, \tag{24}$$

which is real with $|f| < 1$ for $\psi = \pi$ and $\phi > -\ln 2$. The free surface is defined by the equations

$$\frac{dx}{d\phi} = -f = 1 - e^{-\phi}, \qquad \frac{dy}{d\phi} = -\sqrt{1 - f^2} = -\sqrt{2e^{-\phi} - e^{-2\phi}} \qquad (\phi > -\ln 2, \psi = \pi), \tag{25}$$

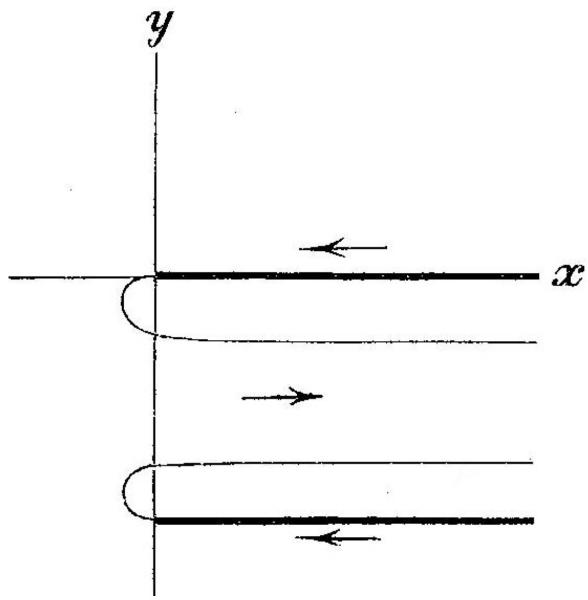
which integrates to

$$x = \phi + e^{-\phi} - 2 + \ln 2, \qquad y = \sqrt{2e^{-\phi} - e^{-2\phi}} - \sin^{-1}(1 - e^{-\phi}) - \frac{\pi}{2} \qquad (\phi > -\ln 2, \psi = \pi), \tag{26}$$

taking the free surface to begin at the origin, and using Dwight 380.311 and 380.001 after the substitution $s = e^{-\phi}$. On the portion of the streamline with $\phi < -\ln 2$, the derivative dz/dw

is purely real, so $dy/d\phi = 0$ and y stays constant at zero; this portion of the streamline is the positive x-axis.

The streamline $\psi = -\pi$ defines the other free surface for $\phi > -\ln 2$; for $\phi < -\ln 2$ it defines the other boundary surface, which is parallel to the positive x-axis at $-y_0$, where y_0 is width of the aperture. The streamlines $\psi = \pm\pi$ are illustrated in the figure below by Kirchhoff.[6]



At large x the upper surface of the free jet is at $y = -\pi$; the width of the jet here is $\Delta\psi = 2\pi$, and the lower surface of the jet is at height π above the bottom of the aperture. That is, the aperture has width 4π, and the width of the contracted jet is 1/2 that of the aperture.

The present example, with a pipe that is re-entrant into a large reservoir of fluid, corresponds exactly to the momentum argument of Borda (sec. 2) that the area of the free jet contracts to 1/2 its initial value. Hence, a re-entrant pipe is sometimes called Borda's mouthpiece.

References

[1] E. Torricelli, *Opera Geometrica* (Florence, 1644),
http://physics.princeton.edu/~mcdonald/examples/mechanics/torricelli_opera_44.pdf

[2] J.-C. de Borda, *Sur l'Écoulement des Fluides par les Ouvertures des Vases*, Mém. Acad. Roy. Sci. p. 579 (1766), http://physics.princeton.edu/~mcdonald/examples/fluids/borda_mars_579_66.pdf

[3] I. Newton, *Philosophiæ Naturalis Principia Mathematica* (1686),
http://physics.princeton.edu/~mcdonald/examples/mechanics/newton_principia.pdf

[6] The function $x(\phi)$ found in eq. (26) has a minimum value of $x = \ln 2 - 1 \approx -0.3$ when $\phi = 0$ and $y = 1 - \pi/2 \approx -0.6$. The tangent to the free surface begins in the $-x$-direction (parallel to the planar boundary) and reverses direction.

[4] J.C. Maxwell, *Remarks on the preceding Paper by G.O. Hanlon on the Vena Contracta*, Proc. London Math. Soc. **3**, 6 (1869),
http://physics.princeton.edu/~mcdonald/examples/fluids/hanlon_prlms_3_4_69.pdf

[5] H. Helmholtz, *Über discontinuirliche Flüssigkeits-Bewegung*, Mon. Berlin Akad. Wissen. 215 (1968), http://physics.princeton.edu/~mcdonald/examples/fluids/helmholtz_mbkpawb_215_69.pdf
On Discontinuous Movements of Fluids, Phil. Mag. **36**, 337 (1868),
http://physics.princeton.edu/~mcdonald/examples/fluids/helmholtz_pm_36_337_68.pdf

[6] G. Kirchhoff, *Zur Theorie frier Flüssigkeitsstrahlen*, J. Math. **70**, 37 (1869),
http://physics.princeton.edu/~mcdonald/examples/fluids/kirchhoff_jm_70_4_37_69.pdf

[7] Lord Rayleigh, *Notes on Hydrodynamics*, Phil. Mag. **2**, 441 (1876),
http://physics.princeton.edu/~mcdonald/examples/fluids/rayleigh_pm_2_441_76.pdf

[8] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 1, 3rd ed. (Clarendon Press, 1904), http://physics.princeton.edu/~mcdonald/examples/EM/maxwell_treatise_v1_04.pdf